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## NOTES ON MAGNETO-HYDRODYNAMICS - NUMBER II

DIMENSIONAL CONSIDERATIONS

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## Preface

This note presents a qualitative discussion of the extent and magnitude of magneto-hydrodynamic phenomena by the elementary means of counting dimensionless parameters.

## Dimensional Considerations - Part 1

A simple means of estimating the complexity of a flow problem is by counting the number of dimensionless parameters which enter. In general, we may say that a significant analytical treatment can be made only when there are no dimensionless parameters (e.g., irrotational incompressible flow) or occasionally, with great effort, when there is one dimensionless parameter (compressible nonviscous flow, incompressible viscous flow). If there are several such parameters, one can expect to obtain a general analytic theory only as a collection of many sub-theories, each applicable to a small range of parameter values. A similar situation exists with respect to numerical computations. In general, three-dimensional problems lie within the capacity of existing machines. For every dimensionless parameter which is to be investigated over its whole range of values, the dimensionality of the problem must be reduced by one. Since, in principle, magneto-hydrodynamics contains all of the phenomena of ordinary fluid dynamics and of electromagnetic theory together with their interaction, one can only expect to develop adequate theories in "patches" which when put together will cover the whole range of phenomena. Furthermore, although the possibility of two

fluid theories involving diffusion in some form exists in ordinary fluid dynamics, it is crucial in magneto-hydrodynamics since the two fluids (electrons and ions) have enormously different properties.

It should be remarked that many linear problems, in particular propagation of plane waves, are not so sensitive to the number of dimensionless parameters and can frequently be solved by a sufficient exercise of patience.

It is conventional to adopt three mechanical dimensional units, usually M, L, T (mass, length, time), and possibly an additional electrical unit. This convention is arbitrary and amounts to a separation of physical laws into two groups, one of which is considered more fundamental than the other. For example, it is possible to consider force, mass, and acceleration as independently defined physical quantities which sometimes are observed to satisfy a relation  $F = kma$ ; the dimensions of  $k$  would be  $FT^2/ML$ . It is even possible to consider velocity, length, and time independently defined so as to satisfy the "experimental" relation

$$v = k' \frac{dx}{dt}$$

where  $k'$  has dimensions  $VT/L$ . On the other hand, the law of universal gravitation could be accorded the same importance as is Newton's second law, and we could define

units such that, in

$$F = G \frac{mm'}{r^2} .$$

$G$  is not only numerically one but also dimensionless; only two of  $M$ ,  $L$ ,  $T$  could be taken as independent units. The choice is one of having more fundamental units and more dimensional physical constants (e.g.,  $k$  and  $G$ ) or fewer of each. The practical consequences of such a choice are slight, but it is worth commenting on this possibility.

Consider Maxwell's equations in the form

$$(1) \quad \left\{ \begin{array}{l} \dot{B} + \text{curl } E = 0 \\ -\dot{D} + \text{curl } H = J \\ \text{div } D = q \end{array} \right. .$$

It is not necessary to consider  $\text{div } B = 0$  since it is dimensionally homogeneous. We take this form of Maxwell's equations since it is dimensionally consistent with all conventional systems of electrical units, no matter what disposition is made of the permeability and permittivity of free space. From (1) we obtain the two groups  $(E, B)$  and  $(D, H, J, q)$ . One member of each group can be assigned an arbitrary dimension, and then all other dimensions are specified. There are two dimensionless combinations, e.g.,

$$(2) \quad \left\{ \begin{array}{l} a_1 = ED/BH \\ a_2 = qE/JB \end{array} \right. .$$

There exist three combinations for each of the dimensions



L, T, V, etc. For example, the three velocities could be

$$E/B, H/D, J/q \quad .$$

The interaction of the electromagnetic field with matter is contained (dimensionally) in the statements that under appropriate circumstances,  $qE$  or  $JB$  can be interpreted as force per unit volume and  $ED$  or  $BH$  as energy per unit volume. Any one of these dimensional identifications implies the other three, and reduces the number of basic electromagnetic units from two to one. (There would seem to be a certain elegance and symmetry in keeping two electrical units with  $kqE$  as force per volume ( $k$  is dimensional), but this will not be pressed in view of the large number of already existing unit systems!)

We now see that the dimensionless parameters (2) can be interpreted as the ratio of electric to magnetic energy and electric to magnetic force respectively. In just the same way that each of the dimensions L, T, V is representable by three distinct combinations it is also possible to construct three distinct combinations having every dimension, e.g., force density, energy density, mass density, mass, etc.

Next we introduce the notation

$$(3) \quad \left\{ \begin{array}{l} B/H = \mu \\ D/E = \kappa \\ HE/BD = 1/\mu\kappa = c^2 \end{array} \right. .$$

There are two approaches, as mentioned in the preceding chapter. We can consider  $\kappa$ ,  $\mu$ ,  $c$  to be universal constants, i.e., give them their vacuum values and include polarization charge and current in  $q$  and  $J$ . Insofar as (3) then amounts to a definition of  $\kappa$ ,  $\mu$ , and  $c$ , the dimensional structure is unchanged. However, under certain circumstances, it will be found that  $q$  or  $J$  splits into two components with a constant ratio dependent on the medium; these ratios will be additional dimensionless parameters. The second approach is to consider  $\mu$  and  $\kappa$  to be properties of the medium which, in certain simple cases are constant (dimensionless) multiples of  $\kappa_0$  and  $\mu_0$ , the vacuum values; in this case (3) holds only approximately under certain simple experimental situations. We shall adopt the first approach as being more appropriate to magneto-hydrodynamics. The various unit systems of electromagnetic theory are obtained by choosing a unit (e.g., charge) or by choosing either  $\mu$  or  $\kappa$  to be dimensionless. Numerically, the energy density is taken sometimes as  $\frac{1}{2} ED$  and  $\frac{1}{2} HB$  and sometimes as  $ED/8\pi$  and  $HB/8\pi$ , while several numerical choices for either  $\mu$  or  $\kappa$  are available.

The addition of any external dimensional parameter, e.g., the length of an object, frequency of an impressed oscillation, etc., increases the number of dimensionless parameters to three, since, as was mentioned, from the electromagnetic quantities alone three combinations having

any dimension can be constructed. The same is true of a physical property of the medium, e.g., conductivity.

This method of counting is only suggestive since an object has not only a length but several dimensions, and, in the last analysis, a shape. Even a conductivity which is not a constant also amounts to an infinity of parameters, in principle. Also, in a particular coordinate system, the ratio of one component of a vector to another can enter, e.g.,  $E \cdot B$  as compared with  $|E \times B|$ . However, keeping in mind the goal of this counting procedure, namely, an estimate of the complexity of the system of equations, these complications can be ignored.

Now, turning to fluid dynamics (one fluid) we have as fundamental dimensional quantities velocity,  $u$ , and two thermodynamic coordinates, e.g.,  $p$  and  $\rho$ . We also introduce the sound speed  $a^2 \sim p/\rho$ . In fluid dynamics alone, we have one dimensionless parameter,  $u/a = \text{Mach number}$ , and two each of velocity, mass density, energy density. Inclusion of viscosity provides an additional length (for example) and consequently a total of two each of every mechanical dimension. Addition of an external length (or frequency) then yields a second dimensionless parameter, the Reynolds number. Each additional external parameter or property of the medium (e.g., gravity, heat conductivity) increases the list by one.

Combining the basic electromagnetic quantities (without polarization or conductivity) with the basic one fluid quanti-

ties (no viscosity, heat conductivity, gravitation) yields a system with five dimensionless and six dimensional combinations. In addition to (2) we can take

$$\begin{cases} a_3 = u/a \\ a_4 = p/BH \\ a_5 = u/c \end{cases} .$$

As representative energy densities we might take

$$ED, BH, qEH/J, p, \rho u^2, \rho c^2 \quad .$$

If, in addition, we include polarization (two kinds), electrical and heat conductivity, viscosity, gravitation, one external length and one frequency, we obtain thirteen dimensionless parameters! There certainly exist problems in which every one of these effects is important, but to consider more than one or two simultaneously is, in practice, impossible.

A common ("classical") set of magneto-hydrodynamic equations is obtained by dropping viscosity, heat conductivity, gravity, polarization, displacement current,  $qE$  compared to  $J \times B$ , and assuming infinite conductivity (this will be considered in detail later). We end up with two dimensionless parameters, e.g.,  $u/a$  and  $p/HB$ , and three each of velocity, mass density, energy density. There is no length or time unless external. Consequently, there is no additional dimensionless parameter if only a single external length or frequency is involved. We might expect that, in complexity and in wealth of phenomena, this set

of equations would be comparable to ordinary compressible viscous fluid dynamics.

We now give a few examples estimating the magnitudes of some of the dimensionless parameters under practical conditions. First, we assign representative scale lengths to  $D$  and  $H$  which represent the order of magnitude of the distances over which these vectors change appreciably. The orders of magnitude are estimated by

$$q = \text{div } D \sim D/L_D$$

$$J = \text{curl } H \sim H/L_H$$

from which

$$\frac{E^2}{c^2 B^2} = \frac{ED}{BH} \sim \frac{qE}{JB} \frac{L_D}{L_H} .$$

If  $L_D$  and  $L_H$  are comparable in magnitude, then the electric to magnetic energy ratio is comparable to the force ratio. This would be the case, in general, for macroscopic variation in which  $L_D$  and  $L_H$  both are on the order of the dimensions of the system, but might not be the case in a sharp transition such as a boundary layer of some kind. In a good conductor, we shall find that  $E \sim uB$  where  $u$  is the fluid velocity. In this case, the electrical energy is negligible compared to the magnetic energy unless the fluid speed approaches that of light. The same is true of the forces unless  $L_D$  is much smaller than  $L_H$ .

## Dimensional Considerations - Part 2

### 1. Introduction

The interaction of an electromagnetic field with a conducting fluid can occur under a great variety of conditions; these may range, for example, from the highly rarefied gases of large extent existing in interstellar space to mercury in a laboratory-sized container. Obviously, in order to cover such a wide range of phenomena, the equations which describe them will have to be of a very general nature. In Part 1 it was mentioned that we cannot hope to solve these equations in their full generality and that any theory will have to be made up of "patches," each one valid over a small range. The purpose of this part of the note is to show explicitly what these "patches" are, i.e., to show when certain terms in the equations have to be retained and when they may be neglected. In order to do this, the equations presented in MH-I\* will be repeated here since they are of sufficient generality to account for most hydromagnetic phenomena. From them, some lengths and times which have fundamental physical significance will be obtained. Finally, a comparison of the various terms in the equations will be made on the basis of these fundamental quantities.

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\* Refers to Notes on Magneto-Hydrodynamics - Number I.

## 2. Basic Equations

We repeat here the equations of MH-I. A conducting fluid, interacting with an electromagnetic field, is governed by the equations of conservation of mass, momentum and energy together with Maxwell's equations, and two relations between the pressure  $p$ , the density  $\rho$ , the temperature  $T$  and the internal energy/unit volume,  $e$  for each fluid. For a perfect gas such as will be assumed here, these two relations are  $p = \rho RT$  and  $e = C_V \rho RT$ ,  $C_V$  being the specific heat at constant volume and  $R$ , the gas constant, possibly different from gas to gas. The equations of interest are the following:

### Conservation of mass

$$(1) \quad \frac{\partial \rho_r}{\partial t} + \frac{\partial}{\partial x^j} (\rho_r u_r^j) = 0$$

### Conservation of momentum

$$(2) \quad \frac{\partial}{\partial t} (\rho_r u_r^j) + \frac{\partial}{\partial x^j} (\rho_r u_r^i u_r^j + P_r^{ij}) = X_r^i$$

### Conservation of energy

$$(3) \quad \frac{\partial \bar{e}_r}{\partial t} + \frac{\partial}{\partial x^j} (\bar{e}_r u_r^j + u_r^i P_r^{ij} + q_r^j) = \mathcal{E}_r$$

### Current equation

$$(4) \quad \frac{\partial J_r^i}{\partial t} + \sum_r \frac{\partial}{\partial x^j} (q_r u_r^i u_r^j + \gamma_r P_r^{ij}) = \sum_r \gamma_r X_r^i$$

We recall that the superscripts refer to the coordinates and the subscripts to the components ( $r = 1, \dots, n$  for an  $n$ -component system)

$$u_r^i = \text{fluid velocity}$$

$$\bar{e}_r = \text{total energy/unit volume (internal + flow)}$$

$$J^i = \sum q_r u_r^i = \text{current observed in fixed system}$$

$$q_r = \text{charge density (signed)} = e_r v_r$$

$$e_r = \text{charge (signed)}$$

$$v_r = \text{number density}$$

$$\gamma_r = \frac{e_r}{m_r}$$

$$m_r = \text{mass}$$

$$p_r^{ij} = \text{stress tensor}$$

$$Q_r^i = \text{heat flow vector}$$

$$X_r^i = \text{force per unit volume}$$

$$\mathcal{E}_r = \text{energy source/unit volume}$$

$X_r$  and  $\mathcal{E}_r$  are assumed to have the following forms:

$$X_r = q_r(E + u_r \times B) - \sum_s a_{rs}(u_r - u_s)$$

$$\mathcal{E}_r = u_r[q_r(E + u_r \times B) - \sum_s a_{rs}(u_r - u_s)]$$

where  $a_{rs}$  represents the "coefficient of friction" between the  $r^{\text{th}}$  and  $s^{\text{th}}$  component. For a two-component system,  $a_{rs} = a_{sr} = a$  is true. Without much loss in generality, we can deal with a two-component system.  $E$  and  $B$  are the electric and magnetic field vectors respectively. Taking  $B$  in the direction 1, under



some conditions, the stress tensor has the following form:\*

$$P^{11} = p - \frac{2}{3} \eta \left( 2 \frac{\partial u^1}{\partial x^1} - \frac{\partial u^2}{\partial x^2} - \frac{\partial u^3}{\partial x^3} \right)$$

$$P^{22} = p - \frac{1}{3} \frac{2\eta}{1 + \frac{16}{9} \omega_g^2 \tau^2} \left\{ \left( 2 \frac{\partial u^2}{\partial x^2} - \frac{\partial u^1}{\partial x^1} - \frac{\partial u^3}{\partial x^3} \right) + \right. \\ \left. + \frac{8}{9} \omega_g^2 \tau^2 \left( \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3} - 2 \frac{\partial u^1}{\partial x^1} \right) - 2 \omega_g \tau \left( \frac{\partial u^2}{\partial x^3} + \frac{\partial u^3}{\partial x^2} \right) \right\}$$

$$P^{23} = P^{32} = - \frac{2\eta}{1 + \frac{16}{9} \omega_g^2 \tau^2} \left\{ \frac{1}{2} \left( \frac{\partial u^3}{\partial x^2} + \frac{\partial u^2}{\partial x^3} \right) + \frac{2}{9} \omega_g \tau \left( \frac{\partial u^3}{\partial x^3} - \frac{\partial u^2}{\partial x^2} \right) \right\}$$

$$P^{12} = P^{21} = - \frac{2\eta}{1 + \frac{4}{9} \omega_g^2 \tau^2} \left\{ \frac{1}{2} \left( \frac{\partial u^2}{\partial x^1} + \frac{\partial u^1}{\partial x^2} \right) + \frac{1}{3} \omega_g \tau \left( \frac{\partial u^3}{\partial x^1} + \frac{\partial u^1}{\partial x^3} \right) \right\}$$

where, approximately,  $\tau \propto \frac{3\eta}{2p}$ ,  $\tau$  being the collision interval and  $\eta$ , the coefficient of viscosity. An approximate expression for  $\eta$  is  $\eta \propto \rho \mu L$  where  $L$  is the mean free path. The gyrofrequency  $\omega_g$  is given by the well-known relation  $\omega_g = \frac{eB}{m}$ .

The expression for the best flow vector  $Q$  is\*\*

$$Q = - \frac{K(\text{grad } T - \left| \frac{\omega_g \tau}{B} \right| B \times \text{grad } T)}{1 + \omega_g^2 \tau^2}$$

$K$  is the coefficient of thermal conductivity;

$$K \propto \rho \mu L C_V .$$

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\* Chapman, S. and Cowling, T.G., The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, 1939, p. 338.

\*\* Ibid, page 337.

Although numerical factors have been neglected in the expressions for  $\gamma$  and  $K$ , this will not invalidate the qualitative conclusions about the relative importance of the terms entering into the equations. At most, these factors will make some slight quantitative differences which can be readily calculated to the required accuracy.

Maxwell's equations

$$(5) \quad \text{curl } E = - \frac{\partial B}{\partial t}$$

$$(6) \quad \frac{1}{\mu} \text{curl } B = K \frac{\partial E}{\partial t} + J$$

$$(7) \quad K \text{div } E = q$$

where  $K$  is the dielectric constant,  $\mu$ , the permeability and  $q = \sum q_r$ . Rationalized M. K. S. units have been used.

This completes the set of basic equations.

### 3. Fundamental Lengths and Times

From the momentum (or current) and energy equations and equation (6), we can immediately make a list of fundamental frequencies by dropping the space dependent terms. The following five frequencies have physical significance:

1)  $\omega_p = (\sum \frac{\gamma_r^q}{\kappa})^{1/2}$  = plasma frequency, which may be obtained from the current equation by neglecting all external forces except the electric field.

2)  $\omega_g = \frac{eB}{m}$  = gyrofrequency. This is obtained from the current equation by dropping all external forces except the magnetic field. Both the plasma and gyro-frequencies have been discussed at sufficient length by numerous authors so as to make any further discussion here of their physical significance entirely superfluous.

3)  $\omega_f = \frac{1}{m} \frac{a_{rs}}{v}$  = friction frequency. The significance of this frequency is most readily seen from the momentum equation by retaining only the friction term in the external force expression.  $\omega_f^{-1}$  represents essentially an e-folding time for fluid velocity due to frictional resistance.

4)  $\omega_E = \frac{J}{\kappa E}$  as obtained from the energy equation by only retaining the electric field terms. In the special case where Ohm's law assumes the form  $J = \sigma E$ ,  $\sigma$  being

the electrical conductivity in the absence of a magnetic field,  $\omega_E$  takes the form  $\omega_E = \frac{2\sigma}{\kappa}$ . It prescribes the rate at which fluid energy is converted into electric energy, for example. This same frequency may be obtained from equation (6) by dropping the space dependent term. There its interpretation is the (e-folding time)<sup>-1</sup> of the electric field.

5)  $\omega_B = c^2 \sigma \mu$  is obtained in a similar way by retaining only the magnetic field in the force term of the energy equation and assuming the special form  $J = \sigma E$  for Ohm's law;  $c$  is the velocity of light in free space.

This completes the list of frequencies. Turning now to velocities, there are three which are of special interest. They are

$c$  = the speed of light

$a = \sqrt{\frac{dp}{d\rho}}$  = speed of sound

$a_{\text{Alf}} = \sqrt{\frac{B^2}{\rho}}$  = speed of Alfvén wave.

Combining these three velocities with each of the five frequencies leads to fifteen lengths which will be listed here

$$1) \quad \frac{a}{\omega_p} = \lambda_D = \text{Debye length}$$

$$2) \quad \frac{c}{\omega_p} = \lambda_D \frac{c}{a}$$

$$3) \quad \frac{a_{\text{Alf}}}{\omega_p} = \lambda_D \frac{a_{\text{Alf}}}{a}$$

$$4) \quad \frac{a}{\omega_g} = \lambda_g \quad \text{where } \lambda_g = \text{mean radius of gyration} .$$

$$5) \quad \frac{c}{\omega_g} = \lambda_g \frac{c}{a}$$

$$6) \quad \frac{a_{\text{Alf}}}{\omega_g} = \lambda_g \frac{a_{\text{Alf}}}{a}$$

$$7) \quad \frac{a}{\omega_f} = \frac{L}{M} \quad \text{where } L \text{ is the mean free path for collisions and } M \text{ (the Mach number)} = \frac{u}{a}$$

From elementary kinetic theory,  $L \propto \frac{1}{\nu S}$ ,  $S$  being the collision cross-section. Furthermore, by straightforward application of dimensional analysis we know that  $\omega_f \propto \nu S u$ . Combining these results yields the above relation for  $\frac{a}{\omega_f}$ .

$$8) \quad \frac{c}{\omega_f} = \frac{L}{M} \frac{c}{a}$$

$$9) \quad \frac{a_{\text{Alf}}}{\omega_f} = \frac{L}{M} \frac{a_{\text{Alf}}}{a}$$

$$10) \quad \frac{a}{\omega_E} = \delta_E \frac{a}{c} \quad \text{where } \delta_E \text{ is the skin depth when the electric field has frequency } \omega_E .$$

The skin depth  $\delta_E$  is given by\*  $\delta_E = \left( \frac{2}{\omega_E \sigma \mu} \right)^{1/2}$  and  $\sigma \propto \alpha \epsilon^2 L \nu / \mu$ \*\*  $\alpha$  being the degree of ionization.

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\* Stratton, J. A., Electromagnetic Theory, the McGraw-Hill Book Company, 1941, (p. 504).

\*\* Elasser, W. M., "Dimensional Relations in Magneto-Hydrodynamics," Physical Review (2) 95: 1-5, 1954.

$$11) \quad \frac{c}{\omega_E} = \delta_E$$

$$12) \quad \frac{a_{\text{Alf}}}{\omega_E} = \delta_E \frac{a_{\text{Alf}}}{c}$$

$$13) \quad \frac{a}{\omega_B} = \delta_B \frac{a}{c} \quad , \quad \delta_B \text{ being the skin depth when the magnetic field has the frequency } \omega_B$$

$$14) \quad \frac{c}{\omega_B} = \delta_B$$

$$15) \quad \frac{a_{\text{Alf}}}{\omega_B} = \delta_B \frac{a_{\text{Alf}}}{c}$$

#### 4. Comparison of Terms in Maxwell's Equations

Starting with equation (6), we inquire under what conditions  $\kappa \frac{\partial E}{\partial t} \ll J$ , i.e., when is the displacement current negligible in comparison with the conduction current? Expressing  $J$  by equation (4) where the space dependent terms have been dropped, we obtain the following condition:  $t \gg \left( \frac{1}{\omega_p^2 + \omega_g^2 + \omega_f^2} \right)^{1/2}$  where

$t$  is the time over which  $E$  begins to vary significantly. The conditions under which  $\text{curl } B \ll J$  have been treated by Elasser\* in full detail and hence will not be discussed here. Likewise, the conditions under which the electrical forces are small compared to the magnetic forces have been treated by Elasser and others. On the other hand, the effects of the various components of the force term in the current equation have not been fully analyzed. For instance

$$\sum_r \sum_s a_{rs} (u_r - u_s) \ll \sum_r q_r u_r \times B$$

when  $\omega_f \ll \omega_g$  (we assume a two-component system where  $a_{rs} = a_{sr}$  and use a coordinate system such that  $u_s = 0$ ).

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\* Ibid.

## 5. Comparison of Terms in Current and Energy Equations

### 1) Current Equation

We compare the space dependent terms of the current equation with  $\frac{\partial J}{\partial t}$ . These terms will be numbered from (1) to (5).  $t$  is chosen to be successively  $(\omega_g)^{-1}$ ,  $(\omega_f)^{-1}$ ,  $(\omega_p)^{-1}$  since these enter into the current equation. The space dependent terms must be retained under the following conditions:

a) $t = (\omega_g)^{-1}$	<div style="display: flex; flex-direction: column; align-items: center;"> <div>(1) <math>x &lt; \lambda_g M</math></div> <div>(2) <math>x &lt; \frac{\lambda_g}{M}</math></div> <div>(3) <math>x^2 &lt; \frac{\lambda_g L}{D} M</math></div> <div>(4) <math>x^2 &lt; \frac{L^3}{\lambda_g} \frac{M^3}{D}</math></div> <div>(5) <math>x^2 &lt; \frac{M^2 L^2}{D}</math></div> </div>
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where  $x$  represents a distance over which the space dependent terms may vary appreciably and  $D \equiv 1 + 4 M^2 \frac{L^2}{\lambda_g^2}$ .

We note that the third term decreases in importance with increasing mean free path.

b) $t = (\omega_p)^{-1}$	<div style="display: flex; flex-direction: column; align-items: center;"> <div>(1) <math>x &lt; \lambda_D M</math></div> <div>(2) <math>x &lt; \frac{\lambda_D}{M}</math></div> </div>
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	(3)	$x^2 < \frac{\lambda_D^{LM}}{D}$
	(4)	$x^2 < \frac{\lambda_D L^3}{\lambda_g^2} \frac{M^3}{D}$
	(5)	$x^2 < \frac{\lambda_D L^2}{\lambda_g} \frac{M^2}{D}$
c) $t = (\omega_f)^{-1}$	(1)	$x < L$
	(2)	$x < \frac{L}{M^2}$
	(3)	$x^2 < \frac{L^2}{D}$
	(4)	$x^2 < \frac{L^4}{\lambda_g^2} \frac{M^2}{D}$
	(5)	$x^2 < \frac{L^3}{\lambda_g} \frac{M}{D} \cdot$

## 2) Energy Equation

Assuming that the internal energy consists only of kinetic energy, under what conditions does the flow energy become comparable with the internal energy? It is easily seen that the condition is  $\frac{2}{R} C_V = M^2$ . For point molecules,  $C_V = \frac{3}{2} R$ , hence we obtain

$$M^2 \cong 3$$

This means that the flow energy compares with the internal energy at supersonic speeds. In the following, we shall consider subsonic speeds and carry out a similar analysis

as for the current equation with  $t$  successively equal to  $(\omega_E)^{-1}$ ,  $(\omega_B)^{-1}$ ,  $(\omega_f)^{-1}$ .

a)  $t = (\omega_E)^{-1}$

(1)  $x < \delta_E M \frac{a}{c}$

(2)  $x < \delta_E M \frac{a}{c}$

(3)  $x^2 < \delta_E LM^3 \frac{a}{cD}$

(4)  $x^2 < \frac{\delta_E L^3}{\lambda_g^2} M^5 \frac{a}{cD}$

(5)  $x^2 < \frac{\delta_E L^2}{\lambda_g} M^4 \frac{a}{cD}$

(6)  $x^2 < \delta_E LM \frac{a}{cD}$

(7)  $x^2 < \frac{\delta_E L^2}{\lambda_g} M^2 \frac{a}{cD}$

b)  $t = (\omega_B)^{-1}$

- Merely replace  $\delta_E$  by  $\delta_B$  in the above inequalities.

c)  $t = (\omega_f)^{-1}$

(1)  $x < L$

(2)  $x < \beta L$  where  $\beta \geq \frac{3}{2}$

(3)  $x^2 < L^2 \frac{M^2}{D}$

(4)  $x^2 < \frac{L^4}{\lambda_g^2} \frac{M^4}{D}$

(5)  $x^2 < \frac{L^3}{\lambda_g} \frac{M^3}{D}$

(6)  $x^2 < \frac{L^2}{D}$

(7)  $x^2 < \frac{L^3}{\lambda_g} \frac{M}{D}$

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FILED ON "8-1-10-YOUR - C.

[illegible]

